Advanced Macroeconomics : Problem Set 3

MRes Economics, Term I

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Problem 1: Endowment Economy. Consider an endowment economy with one agent. The agent is endowed with a constant amount of resources at each period, i.e. $\omega_t = \omega \forall t$.The agent can trade 1-period loans, which caries a (gross) interest rate R_t: denote with α_t the net asset position of the agent. Finally, the agent maximizes her lifetime utility:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

Question 1. Write the budget constrain of the agent. What is/are the state/s variable/s of this problem?

The budget constrain can be written as:

$$c_t + a_{t+1} = a_t R_t + \omega_t \tag{1}$$

where $w_t = w$ for all periods thus:

$$c_t + a_{t+1} = a_t R_t + \omega \tag{2}$$

The variables of the problem are:

- endogenous state variables: $a_t, \bar{a}$
- exogenous states variables: w_t, R_t
- endogenous control: a_{t+1}, c_t

However we have that $w_t = w$ for every period and as such it is just a constant.

Question 2. Define rigorously a recursive competitive equilibrium for this economy

Initially we start with the general definition of recursive competitive equilibrium for this economy.Which is a set of functions that specify:

- quantities: $G(\bar{a}), g(a, \bar{a})$
- lifetime utility: $V(a, \bar{a})$
- prices: $R(\bar{a})$

such that:

1. $V(a, \bar{a})$ solves equation (3) and $g(a, \bar{a})$ is the associated policy function.
2. Prices are competitively determined: R is determined in equilibrium
3. Consistency is satisfied: $G(\bar{a}) = g(\bar{a}, \bar{a})$

where the Bellman equation is:

$$V(a, \bar{a}) = \max_{c \geq 0, a'} \{u(c) + \beta V(a', \bar{a}')\} \tag{3}$$

subject to:

$$c + a' = aR_t + \omega$$

Where a' is the next period asset and $\bar{a}'$ the next periods aggregate asset.

However since in this economy we have only one agent (it would be the same with homogeneous agents) then and as for every lender we need a borrower we must have that $\bar{a} = 0$ and as such the competitive equilibrium becomes:

- quantities: $g(a)$
- lifetime utility: $V(a)$
- prices: R

such that:

1. $V(a)$ solves equation (4) and $g(a)$ is the associated policy function.
2. Prices are competitively determined: R is determined in equilibrium
3. Consistency is satisfied: $g(0) = 0$

where the Bellman equation is:

$$V(a) = \max_{c \geq 0, a'} \{u(c) + \beta V(a')\} \tag{4}$$

subject to:

$$c + a' = aR_t + \omega$$

Question 3. Derive the Euler Equation of this problem

In order to derive the Euler condition we start from the Bellman equation and replacing the budget constrain:

$$V(a) = \max_{a'} \{u(aR + \omega - a') + \beta V(a')\} \tag{5}$$

and taking the first order condition we get:

$$\frac{\partial B.E.}{\partial a'} = 0 \Rightarrow -u'(aR + \omega - a') + \beta V'(a') = 0 \tag{6}$$

we know that at equilibrium we will are looking for a stationary policy function $a' = g(a)$. Using that we can rewrite the first order condition as:

$$-u'(aR + \omega - g(a)) + \beta V'(g(a)) = 0 \tag{7}$$

and the Bellman (equation 5) as:

$$V(a) = u(aR + \omega - g(a)) + \beta V(g(a)) \tag{8}$$

Using the envelope condition we get:

$$V'(a) = Ru'(aR + \omega - g(a)) \tag{9}$$

From equation (9) by moving it to the next period (as it holds for every period)and keeping in mind that $g(a) = a'$ we get:

$$V'(g(a)) = Ru'(g(a)R + \omega - g(g(a))) \tag{10}$$

Finally using equation (10) we can substitute into the first order condition (eq (7)) to find the Euler condition:

$$-u'(aR + \omega - g(a)) + \beta Ru'(g(a)R + \omega - g(g(a))) = 0 \tag{11}$$

or more simply using $c + a' = aR_t + \omega$ we get:

$$u'(c) = \beta Ru'(c') \tag{12}$$

Question 4. What is the equilibrium value of the asset holding a'? What is the equilibrium value of the gross interest rate R

From the fact that there is only one agent we must have that $\bar{a} = a$ but we know that $\bar{a} = 0$ as for every lender there should be a borrower with an equal amount. Hence we have that $a = 0$. Using this we also get $a' = g(a) = g(0)$. However from consistency we know that $g(0) = 0$ and as such we conclude that:

$$a' = g(a) = 0 \tag{13}$$

To find the equilibrium value for the gross interest rate R we use the results $g(a) = 0$ and $a = 0$ along with equation (11) to get:

$$\begin{aligned} -u'(\omega) + \beta R u'(\omega) &= 0 \Rightarrow \\ R &= \frac{1}{\beta} \end{aligned} \tag{14}$$

Problem 2: Example of Incomplete Market Economy Consider a two-period economy. The agent faces the following decisions: in period $t = 0$ he needs to decide how much to consume and save; in period $t = 1$, he needs to decide how much to consume and how much to work. In period $t = 0$ the representative agent receives an endowment equal to I. The endowment in period $t = 1$ is uncertain, since the wage in the period is stochastic. There are m possible state of the world, i.e.

$$\omega = \{\omega_1, \ldots, \omega_m\} \ ,$$

where $\pi_i = Prob[\omega = \omega_i]$, for $i = 1, \ldots, m$. The utility function is:

$$U = u(c_0) + \beta \sum_{i=1}^{m} \pi_i [u(c_{1i} + v(n_i)]$$

where n denotes labor. Assume that there is a risk-free asset, denoted by a' and priced q such that every unit of a' purchased in period 0 pays 1 unit in period 1, regardless the realization of the stochastic wage. Finally, assume that the utility function is such that: $u(c) = log(c)$ and $v(n) = log(1 - n)$.

Question 1. Write the budget constrain of the agent in period 0 and 1.

The budget constrain in period 0 will be:

$$c_0 + qa' = I \tag{15}$$

and budget constrain in period 1 will be:

$$c_{1i} = a' + w_i n_i, \ \forall i = 1, \ldots, n \tag{16}$$

Question 2. Write the resulting consumer problem.

The consumers maximization problem takes the form:

$$\begin{aligned} U &= \max_{c_0, a', \{c_{i1}\}_{i=1}^m, \{n_i\}_{i=1}^m} u(c_0) + \beta \sum_{i=1}^m \pi_i [u(c_{1i}) + v(n_i)] \\ &= \max_{c_0, a', \{c_{i1}\}_{i=1}^m, \{n_i\}_{i=1}^m} log(c_0) + \beta \sum_{i=1}^m \pi_i [log(c_{1i}) + log(1 - n_i)] \end{aligned} \tag{17}$$

subject to:

$$c_0 + qa' = I \tag{18}$$

$$c_{1i} = a' + w_i n_i, \ \forall i = 1, \ldots, n \tag{19}$$

Question 3. Compute the Euler Equation of the problem.

The simplest way to derive the solution is to substitute the constrains into the maximization problem:

$$U = \max_{a', \{n_i\}_{i=1}^m} log(I - qa') + \beta \sum_{i=1}^m \pi_i [log(a' + w_i n_i) + log(1 - n_i)] \tag{20}$$

Subsequently we take the first order conditions:

$$\frac{\partial U}{\partial a'} = 0 \quad \Rightarrow \quad -\frac{q}{I - qa'} + \beta \sum_{i=1}^m \pi_i \frac{1}{a' + w_i n_i} = 0 \tag{21}$$

$$\frac{\partial U}{\partial n_i} = 0 \quad \Rightarrow \quad \beta \pi_i \left(\frac{w_i}{a' + w_i n_i} - \frac{1}{1 - n_i} \right) = 0, \ \forall i = 1, \ldots, m \tag{22}$$

from the equation (21) and by replacing back c_0, c_{1i} from the budget constrains, we get the Euler equation for the inter temporal consumption substitution:

$$\begin{aligned} \frac{1}{c_0} &= \frac{1}{q} \beta \sum_{i=1}^m \pi_i \frac{1}{c_{1i}} \Rightarrow \\ \frac{1}{c_0} &= R\beta E\left[\frac{1}{c_{1i}}\right] \Rightarrow \\ u'(c_0) &= R\beta E[u'(c_{1i})] \end{aligned} \tag{23}$$

where we used the fact that $R = \frac{1}{q}$ and we denoted with E the expectation over possible state realization of the marginal utility. As such equation (23) looks very similar to the normal Euler equation,having the marginal utility on the LHS, with the difference that on the RHS we have the discounted *expected* marginal utility at period one.

Furthermore from equation (22) we get:

$$\begin{aligned} \frac{w_i}{a' + w_i n_i} - \frac{1}{1 - n_i} &= 0 \Rightarrow \\ -\frac{1}{c_{1i}} w_i &= -\frac{1}{1 - n_i} \Rightarrow \\ -u'(c_{1i}) w_i &= v'(n) \end{aligned} \tag{24}$$

where we have the intra-temporal (for period 1) substitution rule between consumption and leisure (or labor as $l = 1 - n$). On the LHS we have minus the marginal utility of consumption time the wage rate and on the left the marginal utility of leisure. Thus ,as intuitively expected, the lost marginal consumption if we increase marginally leisure and the increase in marginal utility from leisure should be equal in equilibrium.

Question 4. Derive an equation that implicitly links the amount of savings a' to the price of the bond q and the parameters of the model.

To achieve that we use equation (22) to solve for $w_i n_i$:

$$w_i n_i = \frac{w_i - a'}{2} \tag{25}$$

and replace it back to equation (21) to get:

$$\frac{q}{I - a'q} = \beta E \left[\frac{2}{a' + w_i} \right] \tag{26}$$

which is an implicit relationship between a' , q and the parameters of the model as w_i is endogenously given.

Question 5. Considering the answers above, is there complete insurance in this model? Why? Explain.

The model we just analyzed does not entail complete insurance. This fact is initially apparent from equation (23), the Euler condition. The result of equation (23) shows that the agent will evaluate the trade off between present and future based on the *expected* future marginal utility. That in turn implies that even in the optimum asset allocation the agent will be ”vulnerable” to different state realization for w_i.

In a world of complete insurance the agent would have been able to acquire an asset such that the realization in period 1 would have been unconditional on the realization of w_i. Thus to have complete insurance we need the assets to be linked to different possible realizations of the states and thus a number of assets at least as big as the possible states of the world in period 1.

Problem 3: Example of Complete Market Economy Consider the setting in Problem 2, with one difference. Instead of a risk-free asset, we assume that the financial market is characterized by "Arrow-securities"; there are m assets traded in time $t = 0$ and each unit of asset i purchased pays off 1 unit if the realized state in period $t = 1$ is i, and 0 otherwise. Consider the same functional form of the utility function as above.

Question 1. Write the budget constraint of the agent in period 0 and 1.

In period 1 the budget constrain will be almost the same as in the previous case with the incomplete markets:

$$c_{1i} = a'_i + w_i n_i,\ \forall i = 1, \ldots, n \tag{27}$$

however in period 0 there are m traded assets and the budget constrain becomes:

$$c_0 + \sum_{i=1}^{m} q_i a'_i = I\ \forall i = 1, \ldots, n \tag{28}$$

we can also combine the 2 period constrains solving out the asset:

$$\begin{aligned} c_0 + \sum_{i=1}^{m} q_i (c_{1i} - n_i w_i) &= I \Rightarrow \\ c_0 + \sum_{i=1}^{m} q_i c_{1i} &= I + \sum_{i=1}^{m} q_i n_i w_i \end{aligned} \tag{29}$$

where on the LHS we have the two period spending on consumption and on the RHS the sources of income.

Question 2. Write the resulting consumer problem.

Using the consolidated constraint the resulting consumer problem takes the form:

$$\begin{aligned} U &= \max_{c_0, \{c_{i1}\}_{i=1}^m, \{n_i\}_{i=1}^m} u(c_0) + \beta \sum_{i=1}^{m} \pi_i [u(c_{1i}) + v(n_i)] \\ &= \max_{c_0, \{c_{i1}\}_{i=1}^m, \{n_i\}_{i=1}^m} log(c_0) + \beta \sum_{i=1}^{m} \pi_i [log(c_{1i}) + log(1 - n_i)] \end{aligned} \tag{30}$$

subject to:

$$c_0 + \sum_{i=1}^{m} q_i c_{1i} = I + \sum_{i=1}^{m} q_i n_i w_i \tag{31}$$

the constrain in this case is written as only one form of $n+1$ constrains.

Question 3. Compute the Euler Equation of the problem.

The Lagrangian takes the form:

$$\mathcal{L}(\cdot) = log(c_0) + \beta \sum_{i=1}^{m} \pi_i [log(c_{1i}) + log(1-n_i)] + \lambda \left(I + \sum_{i=1}^{m} [q_i(n_i w_i - c_{1i})] - c_0 \right) \tag{32}$$

and the F.O.C's are:

$$\frac{\partial \mathcal{L}}{\partial c_0} = 0 \Rightarrow \frac{1}{c_0} = \lambda \tag{33}$$

$$\frac{\partial \mathcal{L}}{\partial c_{1i}} = 0 \Rightarrow \beta \pi_i \frac{1}{c_{1i}} = \lambda q_i \tag{34}$$

$$\frac{\partial \mathcal{L}}{\partial n_{1i}} = 0 \Rightarrow -\beta \pi_i \frac{1}{1-n_{1i}} = -q_i w_i \lambda \tag{35}$$

from equation (33) and (34) we get:

$$\frac{1}{c_0} = \beta \frac{\pi_i}{q_i} \frac{1}{c_{1i}} \tag{36}$$

and from (34) and (35) we get:

$$-\frac{1}{c_{1i}} w_i = -\frac{1}{1-n_i} \tag{37}$$

Question 4. Interpret the equilibrium conditions derived above.

The equilibrium is described by eq. (36) and eq. (37) and the consolidated budget constrain eq. (31). The first one (eq. 36) is the inter-temporal Euler condition that relates the marginal utility from consumption now to the discounted marginal utility of period 1 weighted by the a ratio of the probabilities to the prices of assets. In essence the higher the price of assets (q_i) in comparison to the risk they ensure (π_i) the lower the amount that the agent will save in period 0. This makes intuitively sence as if the cost of insurance is higher than the related probability the agent will not insure and consumes more in period 0. The second (eq. 37) is identical to the incomplete market where we have the intra-temporal (for period 1) substitution rule between consumption and leisure (or labor as $l = 1-n$). On the LHS we have minus the marginal utility of consumption time the wage rate and on the left the marginal utility of leisure. Thus ,as intuitively expected, the lost marginal consumption if we increase marginally leisure and the increase in marginal utility from leisure should be equal in equilibrium.

Problem 4: Endowment Economy with Two-Agents Assume that the economy is composed of two infinitely lived agent. Agent i derives utility from a given consumption stream $\{c_t^i\}$ as in:

$$U_i(\left\{c_t^i\right\}) = \sum_{t=0}^{\infty} \beta^t u_i(c_t^i),\ i = 1, 2$$

Endowments are stationary:

$$\omega_t^i = \omega^i$$

,$\forall$, $i = 1, 2$. Notice that the two agents might have different utilities and different endowments. As before, the agent can trade 1-period bonds, denoted by a whose unit price is q. Denote as a^i the asset holding of agent i.

Question 1. What is the resource constraint of this economy?

The two individual recourse constrains for agent 1 and 2 are:

$$c_t^1 + qa_{t+1}^1 = w^1 + a_t^1 \tag{38}$$

$$c_t^2 + qa_{t+1}^2 = w^2 + a_t^2 \tag{39}$$

and the economy constrain is:

$$\begin{aligned} c_t^1 + qa_{t+1}^1 + c_t^2 + qa_{t+1}^2 = w^1 + a_t^1 + w^2 + a_t^2 \Rightarrow \\ c_t^1 + c_t^2 + q(a_{t+1}^1 + a_{t+1}^2) = w^1 + w^2 + (a_t^1 + a_t^2) \end{aligned} \tag{40}$$

where from the fact that there are only two agent and $\bar{a} = 0$ we have that both $a_{t+1}^1 + a_{t+1}^2$ and $a_t^1 + a_t^2$ are zero and the equation (40) becomes:

$$c_t^1 + c_t^2 = w^1 + w^2,\ \forall t \tag{41}$$

Question 2. What is the condition for clearing of the asset market?

From the above discution and knowing that in every period for every lender there should be a borrower and vise versa. Thus the sum of asset at each year should be equal to zero. More formally:

$$\sum_{i=1}^{2} a_t^i = \bar{a}_t = 0,\ \forall t \tag{42}$$

Question 3. Define in details the recursive equilibrium of this economy [hint: the problem has an aggregate and an individual state variable]

To answer the question we know that we need an aggregate state variable that will capture the economies evolution of assets. However we know that by definition $\bar{a} = 0$ and thus if we use it as a state variable will not be descriptive for the price of asset q as it will always be zero, however the price of asset will not be constant for sure. The problem is that although the total assets might cancel out the price will depend on the demand and the supply of asset and will make them equal in equilibrium . Another point to note is that in equilibrium each period the assets will always cancel out and thus as there are two agents always $a_t^1 = -a_t^2$ as we previously mentioned with the market clearing condition. Thus we need to define a measure of the asset allocation to use as a state variable that the interest rate will depend on. One such measure can be the semi-distance between the assets defined as:

$$d_s = \frac{a_t^1 - a_t^2}{2} = \frac{2a_t^1}{2} = a_t^1 \tag{43}$$

for the second equality we used the market clearing condition, which is true only in equilibrium of the asset market. This is very intuitive result, as by knowing the equilibrium position on one asset the other is immediately defined (as there can be no excess demand in equilibrium). Thus our measure of asset allocation will be d_s and the equilibrium policy function $G(d_s) = d_s'$. Using d_s as the aggregate state we can define the competitive equilibrium as is a set of functions that specify:

- quantities: $G(d_s), g^1(a^1, d_s), g^2(a^2, d_s)$
- lifetime utility: $V^1(a^1, d_s)$,$V^2(a^2, d_s)$
- prices: $q(d_s)$

such that:

1. $V^1(a^1, d_s)$ and $V^2(a^2, d_s)$ solve equation (44) and $g^1(a^1, d_s)$, $g^2(a^2, d_s)$ are the associated policy functions.
2. Prices are competitively determined: q is determined in equilibrium by d_s, the allocation of assets.
3. Consistency is satisfied (for the derivation of the consistency condition look at the next page):

$$\begin{aligned} G(d_s) &= g^1(d_s, d_s), \ \forall d_s \\ -G(d_s) &= g^2(-d_s, d_s), \ \forall d_s \end{aligned}$$

where the Bellman equation is:

$$V^i(a^i, d_s) = \max_{c^i \geq 0, a^{i'}} \left\{ u_i(c^i) + \beta_i V^i(a^{i'}, d_s') \right\}, \; for\ i = 1, 2 \tag{44}$$

subject to:

$$c_t^i + q a_{t+1}^i = w^i + a_t^i, \; for\ i = 1, 2$$
$$d_s' = G(d_s)$$

Where $a^{i'}$ are the next period asset and d_s' the next periods asset allocation.

To derive the consistency condition we start by writing the consistancy condition for agent 1:

$$G(d_s) = g^1(a^1, d_s) \Rightarrow G(d_s) = g^1(a^1, d_s) \tag{45}$$

then we write the consistency for agent 2:

$$G^2(d_s) = g^2(a^2, d_s) \Rightarrow G^2(d_s) = g^2(-a^1, d_s) \Rightarrow G^2(d_s) = g^2(-d_s, d_s) \tag{46}$$

and finally we use the market clearing condition:

$$\begin{aligned} a^1 + a^2 = \bar{a} \Rightarrow \\ a^1 + a^2 = 0 \Rightarrow \\ g^1(a^1, d_s) + g^2(-a^1, d_s) = 0 \Rightarrow \\ G(d_s) + G^2(d_s) = 0 \Rightarrow \\ G^2(d_s) = \end{aligned} \tag{47}$$

Thus (46) becomes:

$$-G(d_s) = g^2(-d_s, d_s) \tag{48}$$